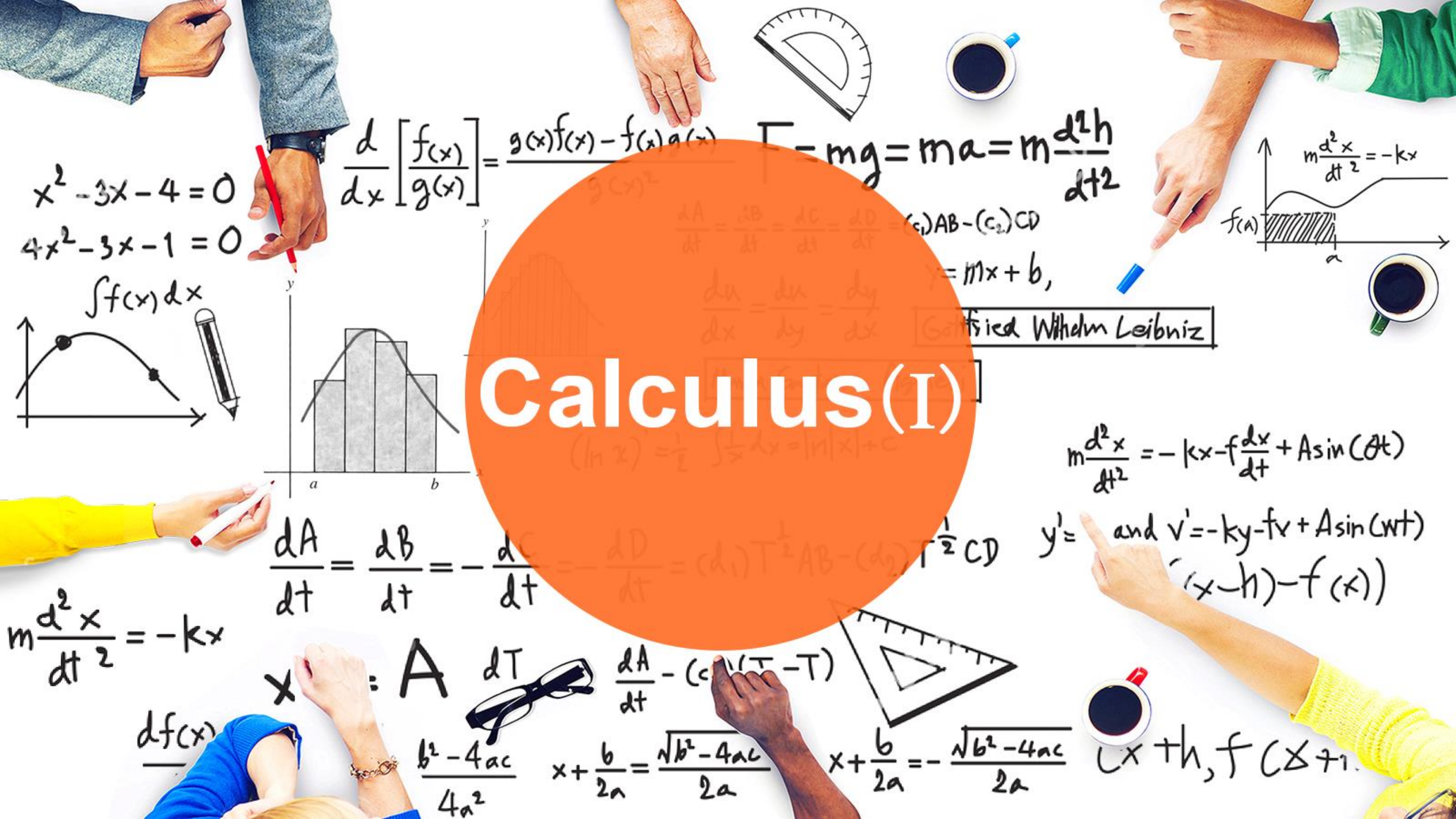
Calculus(I)



3.4 Practical Problems

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In practical life, there are many practical optimization problems, how to deal with the practical problems?

Suggest: step-by step method.

Solving Method: Computation steps

Step1 Draw a picture and assign appropriate variables.

Step2 Write a formula for the objective function Q.

Step3 Express Q as a function of a single variable.

Step4 Find the critical points (end, stationary and singular points).

Step5 Determine the maximum or minimum.

If there is a unique stationary point about the objective function, the function value of this point is the desirable maximum (minimum).

Example 1

Find the greatest volume that a right circular cylinder can have if it is inscribed in a sphere of radius *R*.

Let the height of cylinder be $2h$, radius be r , volume be V

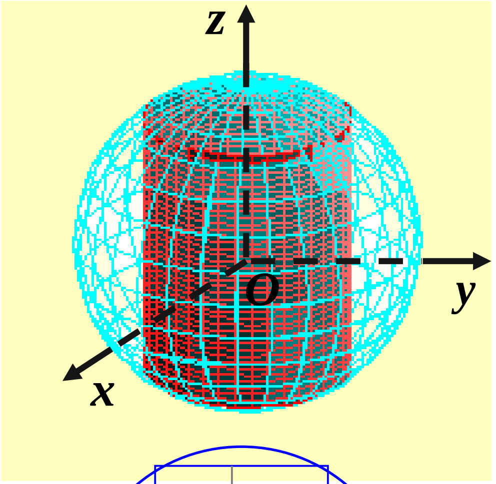
First: Objective function $V = \pi r^2 \cdot 2h$

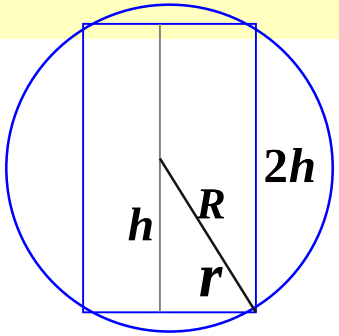
By $r^2 + h^2 = R^2$,

obtain $V = 2\pi(R^2 - h^2) \cdot h$, $0 < h < R$

Then: Find maximum point

$V'_h = 2\pi(R^2 - 3h^2)$





Example 1

Find the greatest volume that a right circular cylinder can have if it is inscribed in a sphere of radius *r*.

$V_h' = 2\pi(R^2 - 3h^2)$

Let $V_h' = 0$, obtain $h = \frac{R}{\sqrt{3}}$ (delete negative value)

(Unique stationary point: the maximum volume of the cylinder must be obtained.)

So the unique stationary point $h = \frac{R}{\sqrt{3}}$ is the maximum value point.

The maximum volume: $V = 2\pi(R^2 - \frac{R^2}{3}) \cdot \frac{R}{\sqrt{3}} = \frac{4\pi}{3\sqrt{3}} R^3.$

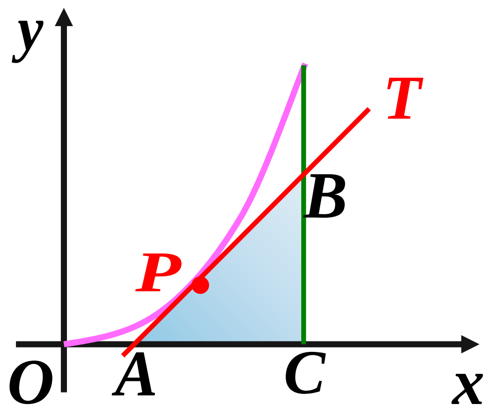
Example 2

The curved triangle area bounded by $\begin{cases} y=0 \\ x=8 \\ y=x^2 \end{cases}$. Find a point $P \in y=x^2$ such that triangle area bounded by the tangent line through P, y=0 and x=8 is the maximum.

As shown, let the tangent point $P(x_0, y_0)$,

the tangent line PT:

$$y - y_0 = 2x_0(x - x_0),$$



$\because y_0 = x_0^2$, $\quad \therefore A(\frac{1}{2}x_0,\ 0)$, $\ C(8,\ 0)$, $\ B(8,\ 16x_0 - x_0^2)$

Example 2

$$\therefore S_{\Delta ABC}=\frac{1}{2}(8-\frac{1}{2}x_0)(16x_0-x_0^2)=x_0(8-\frac{x_0}{2})^2 \quad (0<x_0<8)$$

$$S'_{x_0}=\left(8-\frac{x_0}{2}\right)^2-x_0\left(8-\frac{x_0}{2}\right)=\left(8-\frac{x_0}{2}\right)\left(8-\frac{3x_0}{2}\right)$$

Let $S'_{x_0}=0$, obtain $x_0=\frac{16}{3}$, $x_0=16$ (delete).

Unique stationary point

The maximum area is obtained, so point $P\left(\frac{16}{3},\left(\frac{16}{3}\right)^2\right)$ is the final result.

So $S(\frac{16}{3})=\frac{4096}{27}$ is the maximum area.

Summary of Practical Problems

The steps to solve the practical optimization problems:

- **Step1** Draw a picture and assign appropriate variables.
- **Step2** Write a formula for the objective function Q.
- **Step3** Express Q as a function of a single variable.
- **Step4** Find the critical points (end, stationary and singular points).
- **Step5** Determine the maximum or minimum.

Questions and Answers

A rectangular box is to be made from a piece of cardboard 24 inches long and 9 inches wide by cutting out identical squares from the four corners and turning up the sides.

Find the dimensions of the box of maximum volume.

What is this volume? (See book P167)

Questions and Answers

Let x be the width of the square to be cut out and V the volume of the resulting box .

Then $V = x(9-2x)(24-2x) = 216x - 66x^2 + 4x^3 \left(0 \le x \le 4.5\right)$

$V_x' = 216 - 132x + 12x^2 = 12(x-9)(x-2)$

Let $V_x' = 0$, obtain $x = 2, x = 9$(delete)

So there are only there critical point: 0 , 2 and 4.5 .

$V(0) = 0$, $V(4.5) = 0$, $V(2) = 200$

Then $x = 2$ is the final result. The volume is 200.

The box is 20 inches long, 5 inches wide, and 2 inches deep.

Practical Problems